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A SHARP ISOPERIMETRIC BOUND FOR CONVEX BODIES

BY

RAVI MONTENEGRO*

School of Mathematics, Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
e-mail: monteneg@math.gatech.edu

ABSTRACT

We consider the problem of lower bounding the Minkowski content of subsets of a convex body with a log-concave probability measure, conditioned on the set size. A bound is given in terms of diameter and set size, which is sharp for all set sizes, dimensions, and norms. In the case of uniform density a stronger theorem is shown which is also sharp.

1. Introduction

It is a classical result that among all surfaces in $\mathbb{R}^3$ enclosing a fixed volume, the sphere has minimal surface area, as measured by the Minkowski content μ^+. A related extremal problem shows that half spaces minimize surface area for a Gaussian distribution in $\mathbb{R}^n$ [3, 9].

One variation on these results is to consider surfaces of sets given a log-concave distribution supported on a convex body K, i.e. a closed, bounded convex set with non-empty interior. Recall that the Minkowski content $\mu^+(S) = \lim_{h\to 0^+} \mu(S_h \setminus S)/h$, where $S_h = \{y \in K : \exists x \in S, \|y-x\| \leq h\}$ denotes the set of points at most distance h from S as measured by some norm $\|\cdot\|$. Norms will be assumed to satisfy the property that $\forall c \in \mathbb{R}, u \in \mathbb{R}^n : \|cu\| = |c|\|u\|$.

Our main result is the following.

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THEOREM 1.1: *Let μ be a log-concave probability measure supported on a convex body $K \subset \mathbb{R}^n$. For all measurable sets $S \subset K$ with $\mu(S) \leq 1/2$ it follows that*

$$(diam K)\mu^+(S) \geq \mu(S)G(1/\mu(S)), \tag{1}$$

where the diameter $(diam K)$ is measured in some arbitrary norm $\|\cdot\|$, $\mu^+(S)$ is the Minkowski content in that same norm, and $G(1/\mu(S))$ is given by $G(2) = 2$, while for $\mu(S) < 1/2$ then

$$G(1/\mu(S)) = \frac{\gamma^2 e^\gamma}{e^\gamma(\gamma-1)+1}, \tag{2}$$

where $\gamma > 0$ is the unique solution to

$$\mu(S) = \frac{e^\gamma(\gamma-1)+1}{(e^\gamma-1)^2}.$$

If we are interested in an isoperimetric inequality involving the diameter of the support of the measure, then for every $x \in (0, 1/2]$, dimension n and norm $\|\cdot\|$, there exists a log-concave probability measure μ supported on a convex body $K \subset \mathbb{R}^n$, with a subset $S \subset K$ such that $\mu(S) = x$ and the isoperimetric inequality (1) is an equality.

This extends theorems of Dyer and Frieze [5] and Lovász and Simonovits [8] in which they did not condition on set size. Our sharpness also shows that their theorems are tight only when $\mu(S) = 1/2$.

The main tool in our proof is the Localization Lemma of Lovász and Simonovits, which makes it possible to reduce an n-dimensional integration problem into a one-dimensional problem. A unique aspect of our method is that we start with an unknown lower bound, given by $G(1/x)$, proceed to discover which properties $G(1/x)$ must have to apply Localization, and only at the final step, after reducing this to a one-dimensional problem, do we determine the function $G(1/x)$. All other applications of Localization of which we are aware begin with a conjectured lower bound and proceed to show it to be correct. However, by not making any assumptions to begin with we are able to obtain a sharp lower bound which would have been an unlikely initial candidate.

It does not appear possible to write the function $G(1/x)$ in closed form. However, in Corollary 3.1 we show that $G(1/x)$ behaves like $\log(1/x)$, and in particular $2 + \log(1/2x) \leq G(1/x) \leq 2 + 2\log(1/2x)$. It is interesting to note that for graphs with a nice geometric structure, as with the grid $[k]^n$ (see Example 4.3), this suggests that a bound on cutset expansion $\min_{S\subset V} |Cut(S, V \setminus S)|/\min\{|S|, |V| - |S|\}$ and the edge-isoperimetry $\min_{|S|\leq x|V|} |Cut(S, V \setminus S)|/|S|$ are likely to differ by a logarithmic factor of about $\log(1/x)$, where V is the vertex set and $Cut(A, B)$ is the set of edges from set A to set B.

We are also able to apply our methods to the more specific case of the uniform distribution. In Theorem 4.1 we give a bound which is again sharp for every set size x, dimension n and norm $\|\cdot\|$. The main improvement is when x is small, and in Corollary 4.2 it is shown that when $x < 2^{-n}$ then $G(1/x)$ for the uniform distribution behaves like $n/\sqrt[n]{x}$. Example 4.3 shows that, at least for the ℓ_∞ norm, the extremal cases on the hypercube $[0, 1]^n$ are always within a factor $e \approx 2.7$ of the extremal cases on general convex bodies.

In fact, in general, the extremal cases are relatively simple to state. In the log-concave case with set size $\mu(S)$, dimension n and diameter D fixed, then the long thin cylinder $K = [0, D] \times [0, \epsilon]^{n-1}$ with a one-dimensional exponential distribution $\forall \mathbf{x} = (x_1, x_2, \ldots, x_n) \in K : F(\mathbf{x}) = e^{\gamma x_1}/\epsilon^{n-1} \int_0^D e^{\gamma u} du$ contains a subset $S = [0, s] \times [0, \epsilon]^{n-1}$ that is extremal as $\epsilon \to 0^+$, where γ is from Theorem 1.1 and both s and γ are independent of the dimension n. Likewise, when μ is uniform and volume $\mu(S)$, dimension n and diameter D are fixed, then there is a truncated cone K with a subset S that is extremal (the slope of the cone depends on the dimension n).

Since Theorem 1.1 is sharp, then all bounds of the form

$$(diam K)\mu^+(S) \geq f(\mu(S))$$

will follow as corollaries. For instance,

$$(diam K)\mu^+(S) \geq \mu(S)(1 - \mu(S)) \left(4 + \log \frac{1}{4\mu(S)(1 - \mu(S))}\right)$$

strengthens a result of Kannan, Lovász and Montenegro [6]. A different weakening leads to a Gaussian isoperimetric function,

$$(diam K)\mu^+(S) \geq \sqrt{2\pi} I_\gamma(\mu(S)),$$

where $\mu(S) \leq 1/2$ and $I_\gamma(x)$ is the Gaussian isoperimetric function (see equation (10)).

We note that other authors (e.g. [1, 7]) have proven related results in which quantities measuring the well-roundedness of K were fixed, rather than the diameter D, but their results appear to be tight only in asymptotics and not when conditioned on dimensions and volumes $\mu(S)$ as is the case here. For instance, Bobkov [1] used a Prékopa–Leindler inequality to obtain a related result.

THEOREM 1.2: *For every locally Lipschitz function f on $\mathbb{R}^n$ with values in $[0,1]$, for every symmetric bounded convex set $B \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$,*

$$2\int \|\nabla f(x)\|_{B^\circ} d\mu(x) \geq Ent_\mu(f) + Ent_\mu(1-f) + \log \mu(\{x : \|x - x_0\|_B \leq 1\}).$$

In particular, when $B = rC$ for C the symmetric convex unit ball of $(\mathbb{R}^n, \|\cdot\|)$, and f is an appropriate Lipschitz approximation of the indicator function on a measurable set $S \subset \mathbb{R}^n$, then for every $x_0 \in \mathbb{R}^n$ and every $r > 0$,

$$2r\mu^+(S) \geq \mu(S)\log\frac{1}{\mu(S)} + (1-\mu(S))\log\frac{1}{1-\mu(S)} + \log\mu(\{x : \|x - x_0\| \leq r\}).$$

This is not directly comparable with our result because it considers shape (via r) as well as set size $\mu(S)$. However, if r is half the diameter then this becomes

$$(diam K)\mu^+(S) \geq \mu(S)\log\frac{1}{\mu(S)} + (1-\mu(S))\log\frac{1}{1-\mu(S)},$$

and our result is stronger. Of course, r can be chosen more carefully, in which case the bounds are not comparable.

The paper proceeds as follows. In Section 2 we prove Theorem 1.1. Section 3 proves various bounds on the quantity $G(1/x)$. We conclude with the uniform distribution in Section 4.

2. The proof

Before proving Theorem 1.1 we require a few definitions and a lemma.

Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}^+$ is log-concave if $\forall x, y \in \mathbb{R}^n, t \in [0,1]$: $f[tx + (1-t)y] \geq f(x)^t f(y)^{1-t}$, i.e., $\log f$ is a concave function on the support of f. In particular, non-negative concave functions are log-concave.

A measure μ is log-concave if for every compact $A, B \subset \mathbb{R}^n : \mu(tA + (1-t)B) \geq \mu(A)^t \mu(B)^{1-t}$. Borell [4] showed that log-concave measures are induced by log-concave functions, so that μ is log-concave if and only if there is a log-concave function F such that for every measurable $S \subset \mathbb{R}^n : \mu(S) = \int_S F(x)dx$.

A lower semi-continuous function is one which is a limit of a monotone increasing sequence of continuous functions: for example, the characteristic function of an open set, or the negative of the characteristic function of a closed set.

The lemma below is due to Lovász and Simonovits [8]. It says that if a pair of integral inequalities hold in n dimensions then they also hold on an infinitely thin cone, a so-called "needle". This makes it possible to reduce n-dimensional problems to one-dimensional ones.

LEMMA 2.1 (Localization Lemma): *Let g and h be lower semi-continuous Lebesgue integrable functions on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} g(x)dx \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} h(x)dx \geq 0.$$

Then there exist two points $a, b \in \mathbb{R}^n$ and a linear function ℓ: $[0,1] \to \mathbb{R}^+$ such that

$$\int_0^1 \ell(t)^{n-1} g((1-t)a+tb)dt \geq 0 \quad \text{and} \quad \int_0^1 \ell(t)^{n-1} h((1-t)a+tb)dt \geq 0.$$

The proof of Theorem 1.1 will proceed in three steps. First, Localization is used to show that if an n-dimensional counterexample exists then a one-dimensional counterexample also exists. Next, it is shown that the one-dimensional result holds if it holds for a small class of one-dimensional problems. We finish by determining the extreme cases of this simplified form.

THEOREM 2.2: *Let G: $[2,\infty) \to \mathbb{R}^+$ be a function such that $G(x)$ and $xG(1/x)$ are non-decreasing. Then*

$$\frac{d-t}{t}\mu(B) \geq \min\{\mu(S_1),\mu(S_2)\} G\Big(\frac{\mu(K\setminus B)}{\min\{\mu(S_1),\mu(S_2)\}}\Big) \tag{3}$$

holds for every n-dimensional log-concave measure μ with compact support $K \subset \mathbb{R}^n$, partition $K = S_1 \cup B \cup S_2$ into disjoint measurable sets, and scalars $t \leq dist(S_1, S_2) = \inf_{x\in S_1, y\in S_2} \|x-y\|$ and $d \geq diamK = \sup_{x,y\in K}\|x-y\|$ both relative to some norm $\|\cdot\|$, if and only if it holds whenever K is the interval $[0,1] \subset \mathbb{R}$ with the Euclidean norm $\|x-y\| = |x-y|$ and B is a closed set.

Remark 1: The dependence on B in the lower bound can be removed by replacing $d-t$ with d and $K\setminus B$ with K. However, this leads to a slightly weaker result which no longer extends results of [8] in terms of t.

Remark 2: It will be shown that $G(x)$ from Theorem 1.1 will satisfy the one-dimensional conditions of Theorem 2.2. Setting $B = S_t \setminus S$ and taking $t \to 0^+$ in equation (3) then gives Theorem 1.1.

Proof: The only if condition is necessary as the one-dimensional problem is just a case of the general problem. It remains to show the converse.

Assume a contradiction, i.e. equation (3) always holds for the one-dimensional problem but there is an n-dimensional example with

$$\frac{d-t}{t}\frac{\mu(B)}{\mu(K\setminus B)} < \frac{\min\{\mu(S_1),\mu(S_2)\}}{\mu(K\setminus B)} G\Big(\frac{\mu(K\setminus B)}{\min\{\mu(S_1),\mu(S_2)\}}\Big). \tag{4}$$

If S_1 and S_2 are increased by an ϵ distance, with $B = K\setminus(S_1\cup S_2)$ and $t \leftarrow t-2\epsilon$ decreasing accordingly, then this still gives a counterexample for ϵ sufficiently small. Therefore it can be assumed that S_1 and S_2 are open and B is closed.

To reduce this to a one-dimensional problem first let F denote the density of μ, that is let F be the log-concave function such that $\mu(A) = \int_A F(x)dx$, and let

$$g(x) = F(x)(\alpha\chi_{S_1}(x) - \chi_B(x)) \quad \text{where } \alpha = \mu(B)/\mu(S_1)$$

and

$$h(x) = F(x)(\beta\chi_{S_2}(x) - \chi_B(x)) \quad \text{where } \beta = \mu(B)/\mu(S_2),$$

where χ_S is the characteristic function of set S. The conditions $\int_{\mathbb{R}^n} g(x)dx \geq 0$ and $\int_{\mathbb{R}^n} h(x)dx \geq 0$ both reduce to $\mu(B) - \mu(B) \geq 0$, which is clearly valid.

The Localization Lemma implies that there exist $a, b \in \mathbb{R}^n$ and $\ell\colon [0,1] \to \mathbb{R}^+$ such that $\int_0^1 \ell(u)^{n-1}g((1-u)a+ub)du \geq 0$ and $\int_0^1 \ell(u)^{n-1}h((1-u)a+ub)du \geq 0$.

To put these in the language of a one-dimensional problem, define a one-dimensional measure ν on $K' = [0,1]$ by $\nu(A) = \int_A F((1-u)a+ub)\ell(u)^{n-1}du$. This measure is induced by the function $F((1-u)a+ub)\ell(u)^{n-1}$, which is log-concave as

$$\log(F((1-u)a+ub)\ell(u)^{n-1}) = \log F((1-u)a+ub) + (n-1)\log\ell(u)$$

and both terms in the sum are concave, F because it is log-concave and $\log\ell(u)$ because the log of a linear function is concave. Let

$$S_1' = \{u \in [0,1] : (1-u)a+ub \in S_1\},$$

with S_2' and B' defined similarly. Then the conditions

$$\int_0^1 \ell(u)^{n-1}g((1-u)a+ub)du \geq 0 \quad \text{and} \quad \int_0^1 \ell(u)^{n-1}h((1-u)a+ub)du \geq 0$$

are equivalent to

$$\frac{\nu(S_1')}{\nu(B')} \geq \frac{\mu(S_1)}{\mu(B)} \quad \text{and} \quad \frac{\nu(S_2')}{\nu(B')} \geq \frac{\mu(S_1)}{\mu(B)}. \tag{5}$$

Without loss assume that $\mu(S_1) \leq \mu(S_2)$ and $\nu(S_1') \leq \nu(S_2')$. Given the conditions of equation (5) it follows that $\nu(B')/\nu(K' \setminus B') \leq \mu(B)/\mu(K \setminus B)$. If $\mu(S_1)/\mu(K \setminus B) \leq \nu(S_1')/\nu(K' \setminus B')$ then equation (4) holds for the measure ν as well, because $xG(1/x)$ is non-decreasing. However, suppose instead that $\mu(K \setminus B)/\mu(S_1) \leq \nu(K' \setminus B')/\nu(S_1')$. Then equation (5) shows $\mu(S_1)/\mu(B) \leq \nu(S_1')/\nu(B')$, and since $G(x)$ is non-decreasing it follows that

$$\frac{d-t}{t} \leq \frac{\mu(S_1)}{\mu(B)} G\Big(\frac{\mu(K \setminus B)}{\mu(S_1)}\Big) \leq \frac{\nu(S_1')}{\nu(B')} G\Big(\frac{\nu(K' \setminus B')}{\nu(S_1')}\Big)$$

and hence equation (4) still holds for ν in this case as well. In general, equation (4) holds for measure ν with $K' = [0,1]$ and sets S_1', S_2', and B'.

The norm $\|\cdot\|$ on $\mathbb{R}^n$ induces a norm on $K' = [0,1]$ by $\forall u, v \in [0,1]$: $\|u - v\|_i = \|((1-u)a + ub) - ((1-v)a + vb)\|$. Then $D = \sup_{x,y \in K'} \|x-y\|_i \leq \sup_{x,y\in K} \|x-y\| \leq d$ and $T = \inf_{x \in S_1', y \in S_2'} \|x-y\|_i \geq \inf_{x \in S_1, y \in S_2} \|x-y\| \geq t$, and so $(d-t)/t \geq (D-T)/T$. However, all norms on a segment are equivalent up to a constant factor; these constants cancel out when taking $(D-T)/T$, so it can be assumed that the norm on K' is standard Euclidean norm on $\mathbb{R}$.

Combining the previous two paragraphs gives a contradiction to the assumption that there are no one-dimensional counterexamples with $K' = [0,1]$, B a closed set and the Euclidean norm. ∎

The general one-dimensional problem will now be reduced to an easier problem.

THEOREM 2.3: *Let G: $[2,\infty) \to \mathbb{R}^+$ be a function such that $G(x)$ and $xG(1/x)$ are non-decreasing. Then*

$$\text{(6)} \qquad \frac{1-t}{t}\mu(B) \geq \min\{\mu(S_1), \mu(S_2)\} G\Big(\frac{\mu(K \setminus B)}{\min\{\mu(S_1), \mu(S_2)\}}\Big)$$

holds for every log-concave measure μ on $K = [0,1]$, partition $K = S_1 \cup B \cup S_2$ into disjoint measurable sets with B a closed set, and scalar $t \leq dist(S_1, S_2) = \inf_{x \in S_1, y \in S_2} |x-y|$, if and only if it holds for every exponential measure $\mu(S) = \int_S e^{\gamma u} du$ and every partition $K = S_1 \cup B \cup S_2$ with $S_1 = [0,s)$, $B = [s, s+t]$ and $S_2 = (s+t, 1]$.

Proof: The only if condition is necessary as this is just a case of the general problem. It remains to show the converse. Some calculations will be easier if G is extended to domain $(0,\infty)$ by letting $G(x) = G(2)$ when $x \leq 2$. Then $G(x)$ and $xG(1/x)$ are still non-decreasing for $x \in (0,\infty)$.

Suppose $S_1 = [0, s)$, $B = [s, s+t]$, $S_2 = (s+t, 1]$ and that equation (6) holds for exponential measures. Given an arbitrary log-concave measure μ, denote its (log-concave) density by F, so that $\mu(A) = \int_A F(u)du$. Then let $h(u) = \alpha + \gamma u$ be the line passing through the points $(s, logF(s))$ and $(s+t, logF(s+t))$, and define $\tilde{F}(u) = e^{h(u)}$ with $\nu(A) = \int_A \tilde{F}(u)du$ being the measure induced by $\tilde{F}$. Concavity of $logF$ implies that $\tilde{F}(u) \leq F(u)$ when $u \in B$ and $\tilde{F}(u) \geq F(u)$ when $u \in S_1 \cup S_2$, and so $\nu(B) \leq \mu(B)$, $\nu(S_1) \geq \mu(S_1)$, $\nu(S_2) \geq \mu(S_2)$ and $\nu(K \setminus B) \geq \mu(K \setminus B)$. But then

$$\begin{aligned}
\frac{1-t}{t}\mu(B) &\geq \frac{1-t}{t}\nu(B) \\
&\geq \nu(K \setminus B)\frac{\min\{\nu(S_1), \nu(S_2)\}}{\nu(K \setminus B)} G\Big(\frac{\nu(K \setminus B)}{\min\{\nu(S_1), \nu(S_2)\}}\Big) \\
&\geq \nu(K \setminus B)\frac{\min\{\mu(S_1), \mu(S_2)\}}{\nu(K \setminus B)} G\Big(\frac{\nu(K \setminus B)}{\min\{\mu(S_1), \mu(S_2)\}}\Big) \\
&\geq \min\{\mu(S_1), \mu(S_2)\} G\Big(\frac{\mu(K \setminus B)}{\min\{\mu(S_1), \mu(S_2)\}}\Big)
\end{aligned}$$

where the second inequality is due to the assumption that equation (6) holds for exponential measures (such as ν), the third inequality follows because $xG(1/x)$ is non-decreasing, and the final inequality is because $G(x)$ is non-decreasing. Then equation (6) holds for μ as well.

Given a general partition $K = S_1 \cup B \cup S_2$ we use a trick of Lovász and Simonovits [8]. Assume equation (6) holds when each set is a single interval, as dealt with in the previous paragraph. Let $[r, s] \subseteq B$ be the leftmost maximal interval in B. If $\mu([0, r)) \geq \mu((s, 1])$, then $S_1 \subseteq (s, 1]$ or $S_2 \subseteq (s, 1]$ and the result then follows from

$$\begin{aligned}
\frac{d-t}{t}\mu(B) \geq \frac{d-t}{t}\mu([r,s]) &\geq \mu((s,1]) G\Big(\frac{\mu([0,1] \setminus [r,s])}{\mu((s,1])}\Big) \\
&\geq \min\{\mu(S_1), \mu(S_2)\} G\Big(\frac{\mu(K \setminus B)}{\min\{\mu(S_1), \mu(S_2)\}}\Big).
\end{aligned}$$

Likewise, the result follows if $\mu((s, 1]) \geq \mu([0, r))$ when $[r, s] \subseteq B$ is the rightmost maximal interval in B. Otherwise, consider the consecutive maximal intervals $[r, s]$ and $[u, v]$ of B such that $\mu([0, r)) \leq \mu((s, 1])$ but $\mu([0, u)) > \mu((v, 1])$. Then $S_1 \subseteq [0, r) \cup (v, 1]$ or $S_2 \subseteq [0, r) \cup (v, 1]$; without loss assume S_1. By the single interval case then

$$\begin{aligned}\frac{1-t}{t}\mu([r,s]) &\geq \mu(K\setminus[r,s])\frac{\mu([0,r))}{\mu(K\setminus[r,s])}G\Big(\frac{\mu(K\setminus[r,s])}{\mu([0,r))}\Big)\\ &\geq \mu(K\setminus[r,s])\frac{\mu([0,r)\cap S_1)}{\mu(K\setminus[r,s])}G\Big(\frac{\mu(K\setminus[r,s])}{\mu([0,r)\cap S_1)}\Big)\\ &\geq \mu([0,r)\cap S_1)G\Big(\frac{\mu(K\setminus B)}{\mu(S_1)}\Big)\end{aligned}$$

because $G(x)$ and $xG(1/x)$ are non-decreasing. Likewise,

$$\frac{1-t}{t}\mu([u,v]) \geq \mu((v,1]\cap S_1)G\Big(\frac{\mu(K\setminus B)}{\mu(S_1)}\Big).$$

Adding these expressions together gives

$$\begin{aligned}\frac{1-t}{t}\mu(B) &\geq \frac{1-t}{t}(\mu([r,s])+\mu([u,v]))\\ &\geq \mu(S_1)G\Big(\frac{\mu(K\setminus B)}{\mu(S_1)}\Big).\end{aligned}$$

If $\mu(S_1)\leq\mu(S_2)$ then equation (6) follows, while if $\mu(S_1)>\mu(S_2)$ then

$$\begin{aligned}\mu(S_1)G\Big(\frac{\mu(K\setminus B)}{\mu(S_1)}\Big) &= \mu(K\setminus B)\frac{\mu(S_1)}{\mu(K\setminus B)}G\Big(\frac{\mu(K\setminus B)}{\mu(S_1)}\Big)\\ &\geq \mu(K\setminus B)\frac{\mu(S_2)}{\mu(K\setminus B)}G\Big(\frac{\mu(K\setminus B)}{\mu(S_2)}\Big)\\ &= \mu(S_2)G\Big(\frac{\mu(K\setminus B)}{\mu(S_2)}\Big)\end{aligned}$$

and equation (6) still holds. ∎

We now find the optimal $G(x)$ for the simplified one-dimensional problem, which leads to the optimal function $G(x)$ in Theorem 2.2 as well.

THEOREM 2.4: *Consider a partition of the space $K=[0,1]\subset\mathbb{R}$ into three intervals, $S_1=[0,s)$, $B=[s,s+t]$ and $S_2=(s+t,1]$, and a measure $\mu(S)=\int_S e^{\alpha u}du$ for some fixed $\alpha\in\mathbb{R}$ for every measurable $S\subset[0,1]$. Then,*

$$(7)\qquad \frac{1-t}{t}\frac{\mu(B)}{\mu(K\setminus B)} \geq \frac{\min\{\mu(S_1),\mu(S_2)\}}{\mu(K\setminus B)}G\Big(\frac{\mu(K\setminus B)}{\min\{\mu(S_1),\mu(S_2)\}}\Big)$$

where $G\colon[2,\infty)\to\mathbb{R}$ is defined by $G(2)=2$ and

$$(8)\quad \forall\gamma>0: G(1/x)=\frac{\gamma^2e^\gamma}{e^\gamma(\gamma-1)+1}\quad \text{where } x=\frac{e^\gamma(\gamma-1)+1}{(e^\gamma-1)^2}\in(0,1/2).$$

Moreover, the function $G(1/x)$ is the best possible at every value of x.

Proof: If $\alpha = 0$ then $\mu(B) = t$ and $\mu(K \setminus B) = 1 - t$, so

$$\frac{1-t}{t}\frac{\mu(B)}{\mu(K\setminus B)} = 1 = \sup_{x\in(0,1/2]} xG(1/x)$$

and the bound follows. If $\alpha < 0$ then reverse orientation and consider the interval $[1, 0]$, which reduces the problem to the case $\alpha > 0$. Also, if $\mu(S_2) < \mu(S_1)$ then observe that the same value $x = \mu(S_2)/\mu(K \setminus B) \in (0, 1/2]$ can also be obtained by intervals $S_1' = [0, s')$, $B' = [s' + t]$, $S_2' = (s' + t, 1]$ with $\mu(S_1') < \mu(S_2')$ and $x = \mu(S_1')/\mu(K \setminus B')$. But then $s' < s$, and hence $\mu(B') = \int_{s'}^{s'+t} e^{\alpha t}dt < \int_s^{s+t} e^{\alpha t}dt = \mu(B)$ as $\alpha > 0$, and so

$$\frac{1-t}{t}\frac{\mu(B')}{\mu(K\setminus B')} < \frac{1-t}{t}\frac{\mu(B)}{\mu(K\setminus B)}.$$

It follows that in order to minimize $\frac{1-t}{t}\frac{\mu(B)}{\mu(K\setminus B)}$ it suffices to consider the case $\mu(S_1) \le \mu(S_2)$ and $\alpha > 0$.

Begin by eliminating the variable s. Let

$$x = \frac{\mu(S_1)}{\mu(K\setminus B)} = \frac{\mu([0,s))}{\mu([0,s)\cup(s+t,1])} = \frac{e^{\alpha s}-1}{e^{\alpha s}-1+e^{\alpha}-e^{\alpha(s+t)}}$$

and solve for $e^{\alpha s}$ to obtain

$$e^{\alpha s} = \frac{1+x(e^{\alpha}-1)}{1+x(e^{\alpha t}-1)}.$$

But then

$$\frac{1-t}{t}\frac{\mu(B)}{\mu(K\setminus B)} = (1+x(e^{\alpha}-1))\frac{1-t}{t}\frac{e^{\alpha t}-1}{e^{\alpha}-e^{\alpha t}} > \alpha x + \frac{\alpha}{e^{\alpha}-1} \tag{9}$$

where the inequality follows from setting $D = e^{\alpha}$ in Lemma 2.5. Now, minimize over all $\alpha > 0$:

$$\frac{\partial}{\partial\alpha}\left(\alpha x + \frac{\alpha}{e^{\alpha}-1}\right) = x + \frac{e^{\alpha}-1-\alpha e^{\alpha}}{(e^{\alpha}-1)^2}.$$

When $\alpha > 0$, this is increasing in α and so the value $\alpha = \gamma > 0$ where $\partial/\partial\alpha = 0$ is an absolute minimum, that is, the minimum occurs at the solution to

$$x = \frac{e^{\gamma}(\gamma-1)+1}{(e^{\gamma}-1)^2} \in (0, 1/2).$$

A solution γ always exists because the fraction on the right is a bijection from $\gamma \in \mathbb{R}_{>0}$ to $x \in (0, 1/2)$. Substituting $\alpha = \gamma$ into equation (9) it follows that

$$xG(1/x) = \gamma x + \frac{\gamma}{e^\gamma - 1} = \frac{\gamma^2 e^\gamma}{(e^\gamma - 1)^2}$$

is a choice of $G(1/x)$ for which equation (7) holds.

In fact, this choice is optimal. Given $x \in (0, 1/2)$ consider the measure $\mu(S) = \int_S e^{\gamma u} du$ for γ as in the theorem, and let $S_1 = [0, s)$ be such that

$$x = \frac{\mu(S_1)}{\mu(K \setminus B)} = \frac{\int_0^s e^{\gamma u} du}{\int_0^s + \int_{s+t}^1 e^{\gamma u} du}.$$

Repeating this for $t \to 0^+$ makes equation (9) an equality, and so this choice of G is sharp at all values of $x \in (0, 1/2)$. ∎

LEMMA 2.5: *If $D > 1$ and $t > 0$ then*

$$\frac{1-t}{t} \frac{D^t - 1}{D - D^t} > \frac{\log D}{D - 1},$$

with the inequality becoming an equality in the limit as $t \to 0^+$.

Proof: Let $D = e^\gamma$ for some $\gamma > 0$. Cross multiplying and simplifying, it suffices to show

$$(1-t)(e^{\gamma t} - 1)(e^\gamma - 1) - \gamma t(e^\gamma - e^{\gamma t}) \geq 0.$$

Plugging in the Taylor series for e^x into the left side and factoring out t or $1 - t$ factors gives

$$\begin{aligned} LHS &= \gamma^2 t(1-t) \sum_{k=0}^\infty \frac{(\gamma t)^k}{(k+1)!} \sum_{k=0}^\infty \frac{\gamma^k}{(k+1)!} - \gamma^2 t \sum_{k=0}^\infty \frac{\gamma^k}{(k+1)!}(1 - t^{k+1}) \\ &= \gamma^2 t(1-t) \left[\sum_{k=0}^\infty \gamma^k \sum_{i=0}^k \frac{t^i}{(i+1)!(k-i+1)!} - \sum_{k=0}^\infty \frac{\gamma^k}{(k+1)!} \sum_{i=0}^k t^i \right] \\ &= \gamma^2 t(1-t) \sum_{k=0}^\infty \gamma^k \sum_{i=0}^k t^i \left[\frac{\binom{k+2}{i+1}}{(k+2)!} - \frac{1}{(k+1)!} \right] > 0, \end{aligned}$$

where the inequality uses that $\binom{k+2}{i+1} = k + 2$ for $i \in \{0, k\}$ and $\binom{k+2}{i+1} > k + 2$ for $i \in \{1, \ldots, k - 1\}$.

The limiting case follows easily from L'Hopital's Rule. ∎

Remark 3: The quantity γ in Theorem 2.4 can be interpreted as the slope of $xG(1/x)$ because

$$\begin{aligned}\frac{d}{dx}[xG(1/x)] &= \frac{d}{d\gamma}\left[\frac{\gamma^2 e^\gamma}{(e^\gamma-1)^2}\right]\left[\frac{dx}{d\gamma}\right]^{-1} \\ &= \frac{\frac{(2\gamma e^\gamma+\gamma^2 e^\gamma)(e^\gamma-1)^2-\gamma^2 e^\gamma 2e^\gamma(e^\gamma-1)}{(e^\gamma-1)^4}}{\frac{(e^\gamma+e^\gamma(\gamma-1))(e^\gamma-1)^2-(e^\gamma(\gamma-1)+1)2e^\gamma(e^\gamma-1)}{(e^\gamma-1)^4}} \\ &= \gamma.\end{aligned}$$

Remark 4: Recall the example of sharpness from the introduction. When $x = 1/2$ then $\gamma = 0$ and $F = 1$ is the uniform distribution, with sharpness when S_1 is half the cylinder, i.e. $S_1 = [0, 1/2] \times [0, \epsilon]^{n-1}$. This is the same as the sharpness result for Dyer and Frieze's [5] version of Theorem 1.1 that does not condition on $\mu(S)$. Similarly, when $\mu(S_1)/\mu(K \setminus B) = \frac{1}{2}$ and $t > 0$ in Theorem 2.2 then $F = 1$ and $S_1 = [0, \frac{1-t}{2}) \times [0, \epsilon]^{n-1}$, $B = [\frac{1-t}{2}, \frac{1+t}{2}] \times [0, \epsilon]^{n-1}$ is sharp. This was the sharp case for Lovász and Simonovits [8]. Hence our bounds match those of both pairs of authors at set size 1/2 and are strictly better on smaller sets.

3. Bounding $G(x)$

Theorem 1.1 gives an optimal bound, but it seems impossible to write $G(x)$ in closed form. We give here a few upper and lower bounds which show that $G(x)$ is essentially logarithmic in x.

COROLLARY 3.1: *If $x \in (0, 1/2]$, then*

$$2x(1-x)\left(2 + \log\frac{1}{4x(1-x)}\right) \geq xG(1/x) \geq x(1-x)\left(4 + \log\frac{1}{4x(1-x)}\right),$$
$$2x(1 + \log(1/2x)) \geq xG(1/x) \geq x(2 + \log(1/2x)),$$

and has limit

$$\frac{G(1/x)}{\log(1/x)} \xrightarrow{x \to 0^+} 1.$$

The first lower bound is a stronger form of a result of Kannan, Lovász and Montenegro [6]. Computer plots show that the absolute error is no more than 0.0051, or at most 0.51% of the [0, 1] range of $xG(1/x)$, and the relative error is no more than 7%.

Another lower bound of interest is

$$(diamK)\mu^+(S) \geq \sqrt{2\pi} I_\gamma(\mu(S)),$$

where $I_\gamma(x)$ is the so-called Gaussian isoperimetric function

$$I_\gamma(x) = \varphi \circ \Phi^{-1}(x), \tag{10}$$

where

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad \text{and} \quad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-y^2/2} dy,$$

which makes its appearance in many isoperimetric results, such as Bobkov's [1]. This lower bound is weaker than the first one in the corollary and so we do not prove it here.

Proof: The first upper bound follows from the second one because $\log(1/2x) \leq 1 - 2x + (1-x)\log(1/4x(1-x))$ when $x \in (0, 1/2]$. Likewise, the second lower bound follows from the first because $2 - 4x + (1-x)\log(1/4x(1-x)) \geq \log(1/2x)$ when $x \in (0, 1/2]$.

The first lower bound is equivalent to the inequality

$$\frac{\gamma^2 e^\gamma}{(e^\gamma - 1)^2} \frac{1}{x(1-x)} - \left(4 + \log\left(\frac{1}{4x(1-x)}\right)\right) \geq 0 \tag{11}$$

where

$$x = \frac{e^\gamma(\gamma - 1) + 1}{(e^\gamma - 1)^2}.$$

This inequality holds at $\gamma \to 0^+$ $(x = 1/2)$, and so in order to show that (11) is non-negative it suffices to show that $\frac{d}{d\gamma}(Eqn.11) \geq 0$, or equivalently that the sign of the derivative is never negative. Multiplying the derivative by a positive function does not affect its sign, so positive factors can be canceled out of the derivative before checking its sign. This differentiation and cancellation of terms can be performed repeatedly; if the final expression is non-negative, and if after each intermediate derivative the value at $\gamma \to 0^+$ was non-negative, then it follows that (11) holds.

Now,

$$\frac{d^4}{d\gamma^4}\left[e^{-\gamma} \frac{d^2}{d\gamma^2}\left[e^{-\gamma} \frac{d^4}{d\gamma^4}\left[e^{-\gamma} \frac{d}{d\gamma} \left[\frac{(e^\gamma(\gamma-1)+1)^2(e^\gamma - (\gamma+1))^2(e^\gamma - 1)}{e^\gamma(\gamma - 2) + (\gamma + 2)} \frac{d}{d\gamma}(Eqn.11)\right]\right]\right]\right] = 1296 e^\gamma.$$

It can be verified that each intermediate derivative was non-negative as $\gamma \to 0^+$, and $1296e^\gamma$ is trivially non-negative, so equation (11) follows by the earlier remarks.

To prove the second upper bound let

$$x = \frac{e^\gamma(\gamma-1)+1}{(e^\gamma-1)^2} \quad \text{for } \gamma > 0.$$

Then

$$\begin{aligned}\frac{d}{d\gamma}[2(1+\log(1/2x)) - G(1/x)] &= -\frac{2}{x}\frac{dx}{d\gamma} - \frac{d}{d\gamma}\frac{\gamma^2 e^\gamma}{e^\gamma(\gamma-1)+1} \\ &= \frac{e^\gamma(e^\gamma(\gamma-2)+(\gamma+2))^2}{(e^\gamma-1)(e^\gamma(\gamma-1)+1)^2} \\ &\geq 0\end{aligned}$$

where the inequality is because $\gamma > 0$. It follows that $2x(1+\log(1/2x)) \geq xG(1/x)$.

For the limiting case

$$\lim_{x\to 0+}\frac{G(1/x)}{\log(1/x)} = \lim_{\gamma\to\infty}\frac{\frac{\gamma^2 e^\gamma}{e^\gamma(\gamma-1)+1}}{\ln(\frac{(e^\gamma-1)^2}{e^\gamma(\gamma-1)+1})} = 1. \qquad ∎$$

4. The uniform distribution

When the distribution F is uniform over K then the results from the previous sections can be strengthened slightly. The proof is similar, but without the reduction from log-concavity to $e^{\gamma t}$. Instead, the extremal cases will be truncated cones $\{\mathbf{x} : \| < x_2, x_3 \ldots, x_n > \| \leq 1 + \gamma x_1\}$, which leads to a more tedious computation.

THEOREM 4.1: *Theorem 1.1 holds for the uniform distribution, but with optimal $G(1/x)$ in dimension 1 given by $xG(1/x) = 1$, and in dimension $n > 1$ given by $G(2) = 2$ and*

$$\forall\gamma > 0: \quad xG(1/x) = \frac{\gamma n}{(1+\gamma)^n - 1}\left[\frac{(1+\gamma)^{n-1}\gamma(n-1)}{(1+\gamma)^{n-1}-1}\right]^{1-1/n}$$

where

$$x = \frac{(1+\gamma)^{n-1}[\gamma(n-1)-1]+1}{[(1+\gamma)^{n-1}-1][(1+\gamma)^n-1]} \in (0, 1/2).$$

How much of an improvement does this give over the log-concave result? By fixing a constant $\hat{\gamma} > 0$ and setting $\gamma = \hat{\gamma}/n$, then as $n \to \infty$ the bound in Theorem 4.1 converges to that of the dimension-free Theorem 1.1, just with $\hat{\gamma}$ in place of γ.

For finite n the main difference is for small values of x. In particular, the limiting cases in Corollaries 3.1 and 4.2 (see below) reveal that when n is fixed and $x \to 0^+$ then $G(1/x)$ for the log-concave case is infinitely smaller than for the uniform case. Therefore the log-concave bound is not a good approximation of the uniform result on small subsets, although the following corollary does show that it is a good approximation when $x > 2^{-n}$.

COROLLARY 4.2: *In Theorem 4.1 the quantity $G(1/x)$ is bounded by*

$$\frac{n}{\sqrt[n]{x}} \geq G(1/x) \geq \frac{1}{2}\frac{n}{\sqrt[n]{x}} \quad \text{when } x \leq 2^{-n},$$
$$2\log_2(1/x) \geq G(1/x) \geq 2 + \log(1/2x) \quad \text{when } x > 2^{-n},$$

and has limit

$$\frac{G(1/x)}{n/\sqrt[n]{x}} \xrightarrow{x \to 0^+} 1.$$

Proof: For the first upper bound consider an n-dimensional unit hypercube $K = [0,1]^n$, with ℓ_∞ norm and subcube $S = [0, x^{1/n}]^n$ embedded in the corner. Then $\mu^+(S) = nx^{1-1/n}$, $\mu(S) = x$ and $diamK = 1$. Therefore,

$$G(1/x) \leq (diamK)\frac{\mu^+(S)}{\mu(S)} = \frac{n}{\sqrt[n]{x}}.$$

For the second upper bound again consider the unit hypercube $[0,1]^n$ with ℓ_∞ norm, but this time consider the k-dimensional subcubes

$$S = [0, x^{1/k}]^k \times [0,1]^{n-k}, \quad \text{for } 1 \leq k < n.$$

Then $\mu(S) = x$, $\mu^+(S) = kx^{1-1/k}$ and minimizing over k implies that $G(1/x) \leq (diamK)\mu^+(S)/\mu(S) = \min_{1 \leq k < n} k/\sqrt[k]{x}$. But $k/\sqrt[k]{x} \leq 2\log_2(1/x)$ when $x \in (2^{-(k+1)}, 2^{-k}]$ and so $G(1/x) \leq \min_{1\leq k<n} k/\sqrt[k]{x} \leq 2\log_2(1/x)$.

When $n = 1$ then the first lower bound is trivial.

When $n > 1$ then

$$\begin{aligned} xG(1/x) &\geq \frac{\gamma n}{(1+\gamma)^n - 1}[x((1+\gamma)^n - 1)]^{1-1/n} \\ &\geq \frac{n}{2}x^{1-1/n} \quad \text{if } \gamma \geq 1. \end{aligned}$$

The formula for x is monotone decreasing in γ, so this implies the lower bound when

$$x \leq x_{\gamma=1} \quad \text{where } x_{\gamma=1} = \frac{2^{n-1}(n-2)+1}{(2^{n-1}-1)(2^n-1)} > \frac{2^{n-1}(n-2)}{2^{n-1}2^n} = \frac{n-2}{2^n}.$$

Then $x_{\gamma=1} > 1/2^n$ for $n \geq 3$, and when $n = 2$ then $x_{\gamma=1} = 1/3 > 1/2^n$ again.

The second lower bound follows from Section 3, as the lower bound for the uniform problem is certainly no smaller than that for the general log-concave.

For the limiting case,

$$\begin{aligned}\lim_{x\to 0^+} \frac{G(1/x)}{n/\sqrt[n]{x}} &= \lim_{\gamma\to\infty} \frac{\gamma n \sqrt[n]{(1+\gamma)^{n-1}-1}\frac{[(1+\gamma)^{n-1}\gamma(n-1)]^{1-1/n}}{(1+\gamma)^{n-1}[\gamma(n-1)-1]+1}}{n\sqrt[n]{\frac{[(1+\gamma)^{n-1}-1][(1+\gamma)^{n}-1]}{(1+\gamma)^{n-1}[\gamma(n-1)-1]+1}}} \\ &= \lim_{\gamma\to\infty} \frac{\gamma}{\sqrt[n]{(1+\gamma)^n - 1}} \left\{ \frac{(1+\gamma)^{n-1}\gamma(n-1)}{(1+\gamma)^{n-1}[\gamma(n-1)-1]+1} \right\}^{1-1/n} \\ &= 1. \quad \blacksquare\end{aligned}$$

It is illustrative to compare these bounds to something that is known exactly.

Example 4.3: Given the n-dimensional hypercube $[0,1]^n$ with uniform distribution $F = 1$ and ℓ_∞ norm, consider $S \subset K$ with faces parallel to the surfaces. Bollobás and Leader [2] studied surfaces of minimal surface area for this problem and found that the extremal sets for fixed x are just the k-dimensional subcubes that were used to determine the upper bounds in Corollary 4.2, i.e.,

$$(diam K)\mu^+(S) \geq \mu(S) \min_{k\in\{1,\ldots,n\}} \frac{k}{\sqrt[k]{\mu(S)}}. \tag{12}$$

Simple calculus shows that $k/\sqrt[k]{x} \geq e\log(1/x)$, with equality at $x = e^{-k}$. Therefore, $(diam K)\mu^+(S) \geq e\mu(S)\log(1/\mu(S))$, which shows that the best logarithmic approximation to (12) is only a factor e larger than the general lower bound of Corollary 4.2. Likewise, the upper bound in the corollary is an upper bound to (12) because it was found by fixing k over certain intervals.

Bollobás and Leader solved the hypercube problem in order to find an edge-isoperimetric inequality on the grid $[k]^n$. The bounds of Corollary 4.2 show that in graphs with a nice geometric structure, such as $[k]^n$, then the cutset expansion and the edge-isoperimetry are likely to differ by a logarithmic factor of inverse set size, as mentioned in the introduction.

5. Remarks

The diameter is often a poor measure of the size of a convex body. For instance, the diameter of $[0,1]^n$ in the standard Euclidean ℓ_2 norm is $\sqrt{n}$, whereas the average distance of a point from the center is much smaller. It would be nice if the methods of this paper could be used to allow conditioning on set sizes in results using measures other than diameter, such as in the theorem of Kannan, Lovász and Simonovits [7],

$$\left(\int |x - x_0|\mu(dx)\right)\mu^+(S) \geq (\log 2)\mu(S)(1 - \mu(S))$$

for probability measure μ. The quantity $\int |x - x_0|\mu(dx)$ measures the average radius of the convex body (or probability distribution) centered at x_0, and replaces the diameter in this paper. A sharp result conditioned on set size would read something like

$$\left(\int |x - x_0|\mu(dx)\right)\mu^+(S) \geq \mu(S)F(1/\mu(S)).$$

However, a problem arises when trying to apply our ideas to this form because the average radius, $\int |x - x_0|\mu(dx)$, may increase when the reduction is made to a one-dimensional problem. This contrasts to the diameter, which is non-increasing. Therefore, it is necessary to "waste" an inequality in the Localization Lemma to hold down this average radius, while two more inequalities would be needed to find $F(1/\mu(S))$. Since Localization only allows for two inequalities then our method fails here.

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